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LETTER TO THE EDITOR

Antipercolation and site–bond Potts model in the $q = 1$ limit

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Abstract. The $q = 1$ limit of a site–bond q-state Potts model is shown to be related to the antipercolation problem, i.e. to the statistics of clusters built of alternately black (occupied) and white (unoccupied) sites.

In the site percolation problem, the sites on a lattice are independently occupied (black) with probability p, or free (white) with probability $1-p$ (Shante and Kirkpatrick 1971, Essam 1972, 1980, Stauffer 1979). A cluster is a set of black sites connected by the lattice edges. This problem may be formulated as the $q = 1$ limit of a Potts model (Potts 1952) with multisite interactions (Giri *et al* 1977, Kunz and Wu 1978, Lubensky 1979, Wu 1982). In the antipercolation problem (Sevšek *et al* 1983) a cluster is a set of alternately black and white sites connected by the lattice edges (figure 1(b)). On alternate lattices, changing the colour of one of the sublattices, we recover a site percolation problem with different site occupation probabilities (p and $1-p$) on the two sub-lattices and the Potts model formulation is trivial. In this letter we present a site–bond Potts model formulation of the antipercolation problem which is valid on any lattice, alternate or non-alternate.

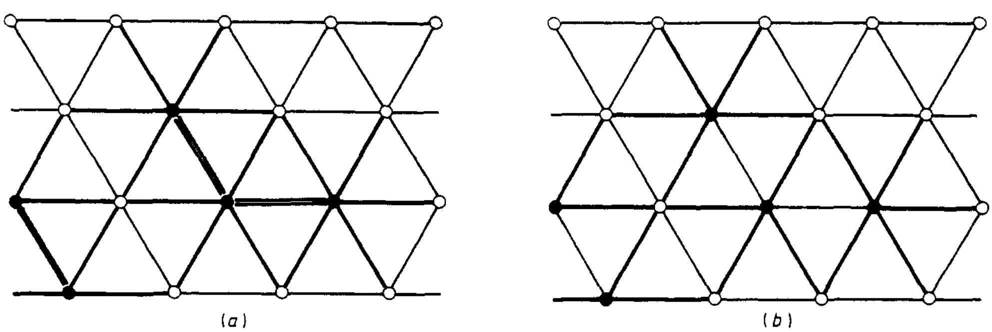

Figure 1. Antipercolation on the triangular lattice: (a) A graph $\mathscr{G}$ on the lattice $\mathscr{L}$; stars with z occupied bonds (heavy line) are centred on black (occupied) sites. First-neighbour sites are linked by a double bond when both are black, a single bond when only one is black; they are disconnected when both are white. (b) Antipercolation cluster corresponding to the graph of figure 1(a).

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Let the q-state site–bond Potts model be defined by its complex Hamiltonian:

$$-\beta\mathscr{H}=\sum_i \ln\left[1-p+p\exp\left(\mathrm{i}\frac{2\pi}{q}\sum_{\substack{j\\ \mathrm{NN}}} r_{ij}(\sigma_i-\sigma_j)\right)\right]+H\sum_i (q\delta_{\sigma_i,0}-1) \tag{1}$$

where the sum over i runs over the N sites of the lattice $\mathscr{L}$, and the sum in the exponential is over the z first neighbours j of site i; a q-state Potts variable $\sigma_i=0,1,\ldots,q-1$ is associated with each site, and another q-state Potts variable $r_{ij}=r_{ji}=0,1,\ldots,q-1$ is associated with each of the $E=Nz/2$ edges of the lattice $\mathscr{L}$. H is an external field favouring the state $\sigma_{i=0}$, δ is a Kronecker delta function and p is the black site occupation probability.

The Hamiltonian possesses the following symmetry:

$$-\beta\mathscr{H}(p)^*=-\beta\mathscr{H}(1-p). \tag{2}$$

The partition function is real and reads

$$\begin{aligned}
Z_N(q;p,H)&\\
&=q^{-E}\operatorname*{Tr}_{\{\sigma,r\}}\exp(-\beta\mathscr{H})\\
&=q^{-E}\exp[(q-1)NH]\operatorname*{Tr}_{\{\sigma\}}\exp\left(\mathrm{q}H\sum_i(\delta_{\sigma_i,0}-1)\right)\operatorname*{Tr}_{\{r\}}\prod_i\\
&\quad\times\left[1-p+p\exp\left(\mathrm{i}\frac{2\pi}{q}\sum_{\substack{j\\ (\mathrm{NN})}} r_{ij}(\sigma_i-\sigma_j)\right)\right]
\end{aligned} \tag{3}$$

which is invariant under the change $p\to 1-p$.

Expanding the last product, to each of the 2^N terms in the expansion corresponds a graph $\mathscr{G}$ on $\mathscr{L}$. A factor $1-p$ corresponds to a white site and a factor p to a black site on $\mathscr{G}$; a factor $\exp[\mathrm{i}(2\pi/q)\sum_{j(\mathrm{NN})} r_{ij}(\sigma_i-\sigma_j)]$ is associated with a star centred on site i with z occupied bonds linking the black site i to its z first neighbours (figure 1(*a*)). When they are first neighbours, black and white sites are connected by a simple bond, whereas two black sites are connected by a double bond and two white sites are disconnected. To a simply occupied edge (ij) corresponds a factor $\exp[\mathrm{i}(2\pi/q)r_{ij}(\sigma_i-\sigma_j)]$, whereas to a free or doubly occupied edge (ij) corresponds a factor 1 since in the last case the exponential enters the product with a plus sign through site i and with a minus sign through site j.

Summing over the edge Potts variables $\{r\}$ and making use of the identity

$$\sum_{r_{ij}=0}^{q-1}\exp\left(\mathrm{i}\frac{2\pi}{q}r_{ij}(\sigma_i-\sigma_j)\right)=q\delta_{\sigma_i,\sigma_j} \tag{4}$$

we get a factor q for each edge and a factor $\delta_{\sigma_i,\sigma_j}$ for each simply occupied edge. The partition function may then be rewritten as

$$Z_N(q;p,H)=\exp[(q-1)NH]\sum_{\mathscr{G}\subseteq\mathscr{L}}P(\mathscr{G})\operatorname*{Tr}_{\{\sigma\}}\exp\left(qH\sum_i(\sigma_{\sigma_i,0}-1)\right)\prod_{(ij)}\delta_{\sigma_i,\sigma_j} \tag{5}$$

where the last product is over the simply occupied edges of $\mathscr{G}$ and

$$P(\mathscr{G})=(1-p)^{N-N_s(\mathscr{G})}p^{N_s(\mathscr{G})} \tag{6}$$

is the probability of occurrence of graph $\mathscr{G}$, $N_s(\mathscr{G})$ the number of occupied sites on $\mathscr{G}$.

Summing over the site Potts variables $\{\sigma\}$ leads to a factor $1+(q-1)\exp(-qHn_s)$ for each connected part (cluster) with n_s sites of $\mathcal{G}$, isolated black or white sites building single site clusters. Equation (5) then becomes

$$Z_N(q;p,H)=\exp[(q-1)NH]\sum_{\mathcal{G}\subseteq\mathcal{L}}P(\mathcal{G})\prod_{n_s}[1+(q-1)\exp(-qHn_s)]^{N\mathcal{K}(\mathcal{G},n_s)} \tag{7}$$

where $N\mathcal{K}(\mathcal{G},n_s)$ is the number of clusters with n_s sites on $\mathcal{G}$.

Let the free energy per site be defined as $(H>0)$

$$f(H)=\lim_{N\to\infty}\lim_{q\to1}\frac{1}{N}\frac{\ln Z_N}{(q-1)}=\lim_{N\to\infty}\frac{1}{N}\frac{\partial\ln z_N}{\partial q}\bigg|_{q=1}$$

$$=H+\sum_{n_s}{}'\,\mathcal{K}(n_s)\exp(-Hn_s) \tag{8}$$

where

$$\mathcal{K}(n_s)=\sum_{\mathcal{G}\subseteq\mathcal{L}}P(\mathcal{G})\mathcal{K}(\mathcal{G},n_s) \tag{9}$$

is the mean number of clusters with n_s sites and the prime indicates that the sum is only over finite clusters since infinite clusters contributions in equation (8) vanish in the thermodynamic limit when $H>0$.

It follows that

$$\lim_{H\to0+}f(H)=\sum_{n_s}{}'\,\mathcal{K}(n_s) \tag{10}$$

is the mean number of finite clusters per site;

$$\lim_{H\to0+}\frac{\partial f(H)}{\partial H}=1-\sum_{n_s}{}'\,\mathcal{K}(n_s)n_s=P^{(a)}(p) \tag{11}$$

is the antipercolation probability, i.e. the probability for any site (either black or white) to belong to an infinite cluster. Finally

$$\lim_{H\to0+}\frac{\partial^2 f(H)}{\partial H^2}=\sum_{n_s}{}'\,\mathcal{K}(n_s)n_s^2=S^{(a)}(p) \tag{12}$$

gives the mean square finite cluster size.

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